

An Application of the First-Order Linear Ordinary Differential Equation to Regression Modeling of Unemployment Rates

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Abstract. The unemployment rate investigates the relationship between labor market outcomes and poverty, evaluates the effect of labor market policies and programs, and provides ways to improve their performance. This study analyzes data-driven regression modeling for the economy, specifically the first-order linear ordinary differential equation (ODE). Consider a collection of actual data for the Ilocos Region's unemployment rate and calculate the numerical derivative. Then, a general equation for the first-order linear ODE is presented, with two parameters that will be determined using regression modeling. Following that, a loss function is defined as the sum of squared errors to reduce the difference between estimated and real data in the presence of fluctuations. After this, a loss function is defined as the sum of squared errors to minimize the differences between estimated and actual data. A set of necessary conditions is derived, and the regression parameters are analytically determined. Based on these optimal parameter estimates, the solution of the first-order linear ODE, which matches the actual data trend, shall be obtained. The observations show that the relationship between the actual data and the adjusted predicted regression dynamics closely matches. Results also indicate that the new insight includes the analysis of fluctuations in the unemployment rate for regression modeling dynamics. This research helps Filipino economists provide insights and inform policy decisions aimed at the labor market, and they can focus their efforts on improving these indicators to stimulate job creation and reduce unemployment.

Keywords: Unemployment rate; Differential equation; Regression modeling; Parameter estimation; Ilocos region; Philippines.

1.0 Introduction

The unemployment rate is a key economic indicator used to assess the health of an economy. It tends to fluctuate with the business cycle, rising during recessions and falling during expansions. It is one of the statistics most closely monitored by policymakers, investors, and the public (Bondarenko, 2024).

In the Philippines, unemployment and underemployment are the most significant challenges and critical indicators of economic problems. According to Rutkowski (2015), poverty is primarily caused by the poor's low earning ability and lack of regular and successful employment opportunities. Behind these are two connected primary causes of in-work poverty: a lack of education among the poor and an absence of effective employment opportunities.

The Public Employment Services Office of Pangasinan, Ilocos Region, announced that the unemployment rate in the province has decreased. Based on the most recent monitoring of PESO Pangasinan as of April 2023, the unemployment rate of the province decreased, with the working population in the province at 1,950,882, or

equal to 60%. 56,097 are among the unemployed, or 4.5% unemployed. 1,246,613 are in the labor force, 1,190,515 are employed, and 158,319 are underemployed, or 12.7%.

Currently, there is a dearth of research regarding modeling the unemployment rate in the Ilocos Region. Although various studies have used the first-order linear ordinary differential equation to help in understanding and optimizing business systems, there has been minimal research into how the model is utilized in the employment sector. The research aimed to develop a new model that can track actual data from historical data, including fluctuations, which would allow it to analyze the trend of the unemployment rate and provide solutions to the problem of economic system development.

The researcher was motivated by the significance of mathematical modeling, particularly in the dynamics of the economic system. Future researchers, even students, and teachers, must be knowledgeable about how mathematical modeling works to help the economic dynamics. Aside from biological ecosystems, the first-order linear ordinary differential equation for the regression model is most commonly applied in business ecosystems, where population symbolizes the sales or revenue of products, technology, channels, or enterprises.

2.0 Methodology

2.1 Research Design

This study employed applied mathematics, specifically focusing on the First-Order Linear Ordinary Differential Equation for Regression Modeling. To achieve the study's objectives, a comprehensive review of related literature was conducted, encompassing various definitions and methodologies for future reference.

2.2 Data Source

Data were gathered from the Philippines Statistics Authority (PSA), Regional Statistics Services Office I, located in the City of San Fernando, La Union. Computer software such as Microsoft Excel and MatLab were utilized to verify calculations and generate graphs to illustrate the actual phenomenon in the regression modeling system.

2.3 Parameter Estimation

Ordinary and partial differential equations are typically used to represent mathematical models of dynamic processes. They encompass both dependent and independent variables, as well as constants known as parameters. Dependent variables are often directly measured, while parameter estimates rely on input and output data. In differential equations, the equation structure and the measurements of the input and initial conditions are known, but some or all of the parameters may be unknown. The objective is to determine the optimal estimate of these parameters to effectively describe many phenomena in real-world systems (Wang and Barber, 2014).

Gaussian process regression (GPR) predicts based on training data, akin to k-Nearest Neighbors. It performs well with small data sets and provides predictions with uncertainty quantification. Prior mean and covariance must be specified, with the covariance defined through a kernel object. The kernel hyperparameters are optimized during fitting by maximizing the log-marginal-likelihood (LML) using the chosen optimizer. Given that this maximization is non-convex (with many local optima), the optimizer needs to be restarted multiple times. The first iteration starts with the original hyperparameters, followed by subsequent iterations using randomly selected hyperparameters from the allowable range (Sit, 2019).

Gaussian processes offer a method for modeling probability distributions over functions. According to Do (2007), they discuss how probability distributions over functions can be utilized within the framework of Bayesian regression.

Let $S = \{(x^{(i)}, y^{(i)})\}_{i=1}^m$ be training set of independent and identically distributed examples from some unknown distribution. In the Gaussian process regression model,

$$y^{(i)} = h(x^{(i)}) + \varepsilon^{(i)}, \quad i = 1, 2, 3, \dots, m$$

where the $\varepsilon^{(i)}$ are independent and identical distributed "noise" variables with independent $\mathcal{N}(0, \sigma^2)$ distributions. Similarly, in Bayesian regression, assume a prior distribution over function $h(\cdot)$ in particular, assume a zero-mean Gaussian process prior,

$$h(\cdot) \sim \mathcal{GP}(0, (k(\cdot, \cdot)))$$

for some valid covariance function $(k(\cdot, \cdot))$.

This study applied Gaussian process regression, a form of machine learning algorithm. Gaussian processes do not aim to find "best-fit" data models or, in general, "best guess" predictions for additional inputs to the test. The researcher makes a trial and error for Gaussian Process Regression codes in MatLab online software to experiment with the parameters to create a second distribution across models or prediction distributions for new test data. These distributions help in measuring uncertainty in model estimations, improving predictions for additional test locations, and comparing the solution to the first-order linear ordinary differential equation.

3.0 Results and Discussion

3.1 The Mathematical Model

Consider a set of actual data, given by

$$y = \{y_1, y_2, y_3, \dots, y_n\}, \quad (1)$$

for time t_i , $i = 1, 2, 3, \dots, n$, where n is the number of data points, and the derivative of data is expressed by

$$y'_i = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}, \quad i = 1, 2, 3, \dots, n-1, \quad (2)$$

Define a loss function

$$\partial J_{se}(\alpha, \beta) = \sum_{i=1}^n (y'_i - \alpha - \beta y_i)^2, \quad (3)$$

with a first-order ordinary differential equation

$$\frac{dy}{dt} = \alpha + \beta y, \quad (4)$$

Where α and β are unknown parameters, and y is the solution set for the first-order linear ODE in system (4). Therefore, this problem is referred to as a regression problem with a first-order linear ODE, and the parameters α and β are known as the regression parameters to be determined later.

Now, consider the gradients of loss function in system (3)

$$\frac{\partial J_{se}}{\partial \alpha} = -2 \sum_{i=1}^{n-1} (y'_i - \alpha - \beta y_i), \quad (5)$$

$$\frac{\partial J_{se}}{\partial \beta} = -2 \sum_{i=1}^{n-1} (y_i)(y'_i - \alpha - \beta y_i), \quad (6)$$

Setting these gradients to zero and doing some algebraic manipulations to gain

$$\hat{\alpha} = \bar{y}' - \hat{\beta} \bar{y}, \quad (7)$$

$$\hat{\beta} = \frac{\sum_{i=0}^n y_i (y'_i - \bar{y}')}{\sum_{i=0}^n (y_i)(y_i - \bar{y})}, \quad (8)$$

Substitute $\hat{\alpha}$ and $\hat{\beta}$ into system (4), we have

$$\frac{dy}{dt} = \hat{\alpha} + \hat{\beta}y, \quad (9)$$

Notice that the analytical solution for system 9 is presented by

$$\hat{y}(t) = \begin{cases} y(\tau)e^{\hat{\beta}(t-\tau)} - \frac{\hat{\alpha}}{\hat{\beta}}[1 - e^{\hat{\beta}(t-\tau)}], & \text{for } t > \tau \\ y(\tau)e^{\hat{\beta}(t-\tau)} + \frac{\hat{\alpha}}{\hat{\beta}}[e^{\hat{\beta}(t-\tau)} - 1], & \text{for } t < \tau \end{cases} \quad (10)$$

for $t > \tau$, where $\hat{y}$ is the estimated point to the actual point in system (1). Thus, system (10) represents the regression model for the actual data in system (1), which closely tracks the actual data trend. This is done for mathematical manipulations purposes (Kek, Chen, and Chan 2024).

3.2 Regression Equilibria Analysis and Existence Conditions

In most mathematical models through differential equations, some equations are nonlinear or contain a higher-order derivative that is difficult and time-intensive to solve (Ming, 2017). According to Khalil (2002), one technique for assessing the model's behavior is to first identify the stability states or places where the rate of the dependent variable with respect to the independent variable(s) equals zero. These equilibria represent the points at which the regression model reaches balance or stability.

For analyzing equilibria in first-order ODEs for regression models, find the values of the independent variables at which the dependent variable's derivative is zero (Braun, 1983). These points provide insights into the regression model's long-term behavior or steady states, which aids in understanding the dynamics and stability of the modeled interaction (Lunt, 2013). Lastly, when dealing with first-order ODEs in regression models, stable points are those where the dependent variable's rate of change is zero, indicating that the variables' connection is balanced or stable (Khalil, 2002).

The following mathematical manipulations below represent possible dynamics of the modified model:

Theorem 1.

(Parameter $\hat{\alpha}$) If the $\frac{\partial J_{se}}{\partial \alpha} = -2 \sum_{i=1}^{n-1} (y'_i - \alpha - \beta y_i)$, then the solution of $\hat{\alpha} = \bar{y}' - \hat{\beta}\bar{y}$.

Proof: At the system (5), suppose that the right-hand side of the equation is equated to zero, and the solution for $\hat{\alpha}$ that will satisfy this equation is determined. Doing so, system (2) becomes

$$-2 \sum_{i=1}^{n-1} (y'_i - \alpha - \beta y_i) = 0$$

This implies that,

$$\begin{aligned} -2 \sum_{i=1}^{n-1} y'_i + (-\alpha) \left(-2 \sum_{i=1}^{n-1} 1 \right) - \left(-2\alpha\beta \sum_{i=1}^{n-1} y_i \right) &= 0 \\ -2 \sum_{i=1}^{n-1} y'_i + 2\alpha \sum_{i=1}^{n-1} 1 + 2\alpha\beta \sum_{i=1}^{n-1} y_i &= 0 \end{aligned}$$

Then,

$$2[-(n-1)\bar{y}' + \alpha(n-1) + \beta(n-1)\bar{y}] = 0$$

According to James (2023) the transition from y'_i to $\bar{y}'$ occurs when moving from the individual data points y'_i to the mean value $\bar{y}$, which can be simplified to

$$\begin{aligned} -2(n-1)\bar{y}' + 2\alpha(n-1) + 2\beta(n-1)\bar{y} &= 0 \\ \left(\frac{1}{2(n-1)}\right)[-2(n-1)\bar{y}' + 2\alpha(n-1) + 2\beta(n-1)\bar{y}] &= 0 \\ -\bar{y}' + \alpha + \beta\bar{y} &= 0 \\ -\alpha &= -\bar{y}' + \beta\bar{y} \\ \hat{\alpha} &= \bar{y}' - \beta\bar{y} \end{aligned}$$

In the third equality, the researcher simplifies the expression, factors out common terms, and often represents the entire summation in a more compact form for this instance. The researcher observed that there is a transition from using the individual term y'_i to the sample mean $\bar{y}'$ between the second and third equalities. Initially, this transition is made to simplify the notation and focus on the average behavior of the system. So, according to Kutner (2005) y'_i refers to the rate of change of each individual data point, while $\bar{y}$ refers to the rate of change to the mean value.

However, according to Larsen and Marx (2012), in the many mathematical manipulations, it is often useful to work with average values rather than individual ones. This is where a sample $\bar{y}'$ comes into play. The sample mean represents the average of the derivatives of data point.

Hence, the solution for intercept α is $\bar{y}' - \beta\bar{y}$. Observe that the difference between the mean of the predicted values of the dependent variable $\bar{y}'$ and the product of the slope term in the linear regression model β and the mean of the observation values of the dependent variable $\bar{y}$ (Fox, 2016).

Theorem 2.

(Parameter $\hat{\beta}$) Show that $\frac{\partial J_{se}}{\partial \beta} = -2 \sum_{i=1}^{n-1} (y_i)(y'_i - \alpha - \beta y_i)$, then $\hat{\beta} = \frac{\sum_{i=0}^n (y_i)(y'_i - \bar{y}')}{\sum_{i=0}^n (y_i)(y_i - \bar{y})}$.

Proof: Evaluating the system (6), the right-hand side of the equation is equated to zero, and the solution for $\hat{\beta}$ that will satisfy this equation is determined, which would give

$$\begin{aligned} -2 \sum_{i=1}^{n-1} (y_i)(y'_i - \alpha - \beta y_i) &= 0 \\ -2 \sum_{i=1}^{n-1} (y_i)(y'_i) + \left[(-2)(-\alpha) \sum_{i=1}^{n-1} y_i \right] + (-2)(-\beta) \sum_{i=1}^{n-1} y_i^2 &= 0 \end{aligned}$$

Then

$$-2 \sum_{i=1}^{n-1} (y_i)(y'_i) + 2\alpha \sum_{i=1}^{n-1} y_i + 2\beta \sum_{i=1}^{n-1} y_i^2 = 0$$

Substituting α to $\bar{y}' - \beta\bar{y}$ leads to

$$\begin{aligned} -2 \sum_{i=1}^{n-1} (y_i)(y'_i) + 2(\bar{y}' - \beta\bar{y}) \sum_{i=1}^{n-1} y_i + 2\beta \sum_{i=1}^{n-1} y_i^2 &= 0 \\ -2 \sum_{i=1}^{n-1} (y_i)(y'_i) + (2\bar{y}' - 2\beta\bar{y}) \sum_{i=1}^{n-1} y_i + 2\beta \sum_{i=1}^{n-1} y_i^2 &= 0 \end{aligned}$$

Simplifying further

$$\begin{aligned} -2 \sum_{i=1}^{n-1} (y_i)(y'_i) + 2 \sum_{i=1}^{n-1} y_i \bar{y}' - 2\beta \sum_{i=1}^{n-1} y_i \bar{y} + 2\beta \sum_{i=1}^{n-1} y_i^2 &= 0 \\ -2 \sum_{i=1}^{n-1} (y_i)(y'_i) + 2 \sum_{i=1}^{n-1} y_i \bar{y}' &= -2\beta \sum_{i=1}^{n-1} y_i^2 + 2\beta \sum_{i=1}^{n-1} y_i \bar{y} \end{aligned}$$

$$-2 \left[\sum_{i=1}^{n-1} (y_i)(y'_i) - \sum_{i=1}^{n-1} y_i \bar{y}' \right] = -2 \left[\beta \sum_{i=1}^{n-1} y_i^2 - \beta \sum_{i=1}^{n-1} y_i \bar{y} \right]$$

By cancellation law

$$\sum_{i=1}^{n-1} (y_i)(y'_i) - \sum_{i=1}^{n-1} y_i \bar{y}' = \beta \sum_{i=1}^{n-1} y_i^2 - \beta \sum_{i=1}^{n-1} y_i \bar{y}$$

According to Montgomery (2021) both summations are now from $i = 0$ to n , which includes all relevant terms or observations. This extension might be done for mathematical convenience or to insure that all data points, including the first one $i = 0$, are considered in the derivation. In the dynamics of regression analysis, this summation typically involves the calculation of the numerator or denominator in the solution for $\hat{\beta}$, where each term corresponds to a specific data point or observation in the data set.

$$\sum_{i=0}^n (y_i)(y'_i) - \sum_{i=0}^n y_i \bar{y}' = \hat{\beta} \left[\sum_{i=0}^n y_i^2 - \sum_{i=0}^n y_i \bar{y} \right]$$

Finally, obtaining

$$\begin{aligned} \hat{\beta} &= \frac{\sum_{i=0}^n (y_i)(y'_i) - \sum_{i=0}^n y_i \bar{y}'}{\sum_{i=0}^n y_i^2 - \sum_{i=0}^n y_i \bar{y}} \\ \hat{\beta} &= \frac{\sum_{i=0}^n (y_i)[(y'_i) - \bar{y}']}{\sum_{i=0}^n (y_i)(y_i - \bar{y})} \end{aligned}$$

The solution for parameter $\hat{\beta} = \frac{\sum_{i=0}^n (y_i)[(y'_i) - \bar{y}']}{\sum_{i=0}^n (y_i)(y_i - \bar{y})}$. Therefore, when substituting $\hat{\alpha}$ and $\hat{\beta}$ into the system (4), the solution of the rate of change of y with respect to time is $\frac{dy}{dt} = \hat{\alpha} + \hat{\beta}y$.

Theorem 3

The modified model admits two (2) possible analytical solutions for $\frac{dy}{dt} = \hat{\alpha} + \hat{\beta}y$. These are

$$\hat{y}(t) = \begin{cases} y(\tau)e^{\hat{\beta}(t-\tau)} - \frac{\hat{\alpha}}{\hat{\beta}}[1 - e^{\hat{\beta}(t-\tau)}], & \text{for } t > \tau \\ y(\tau)e^{\hat{\beta}(t-\tau)} + \frac{\hat{\alpha}}{\hat{\beta}}[e^{\hat{\beta}(t-\tau)} - 1], & \text{for } t < \tau \end{cases}$$

where $\hat{y}$ is the estimated point to the actual point in system (1).

Proof: Suppose that $\hat{\alpha}$ and $\hat{\beta}$ are real numbers

This implies that

$$\begin{aligned} \frac{dy}{dt} &= \hat{\alpha} + \hat{\beta}y \\ dt &= \frac{1}{\hat{\alpha} + \hat{\beta}y} dy \end{aligned}$$

Then applying the indefinite integral both sides leads to

$$\begin{aligned} \int dt &= \int \frac{1}{\hat{\alpha} + \hat{\beta}y} dy \\ t + C &= \frac{1}{\hat{\beta}} \ln |\hat{\alpha} + \hat{\beta}y| \\ \hat{\beta}(t + C) &= \ln |\hat{\alpha} + \hat{\beta}y| \end{aligned}$$

$$\hat{\beta}t + \hat{\beta}C = \ln |\hat{\alpha} + \hat{\beta}y|$$

Since $\hat{\beta}C$ is a constant, it follows that

$$\hat{\beta}t + C = \ln |\hat{\alpha} + \hat{\beta}y|$$

Applying the exponential e to both sides

$$\begin{aligned} e^{\hat{\beta}t+C} &= e^{\ln_e |\hat{\alpha}+\hat{\beta}y|} \\ e^{\hat{\beta}t}e^C &= \hat{\alpha} + \hat{\beta}y \end{aligned}$$

$$Ce^{\hat{\beta}t} = \hat{\alpha} + \hat{\beta}y \quad (11)$$

The expression $Ce^{\hat{\beta}t}$ represents an exponential function where C is a constant and $\hat{\beta}$ is another constant raised to the power of t , a variable (Montgomery, 2021) this function grows or decays exponentially depending on the sign of $\hat{\beta}$. According to Fox (2016) considering that when $\hat{\alpha} + \hat{\beta}y > 0$, the predicted value is positive, this means that it falls above the regression line. Conversely, when $\hat{\alpha} + \hat{\beta}y < 0$, the predicted value is negative, which means it falls below the regression line.

According to Harrell (2015) in regression analysis, time is usually modeled as a predictor variable with the potential to be negative. Time is usually considered as a continuous variable that moves in a positive direction, representing the order of time of observations. However, depending on the point of reference or basis used, associated with time variables in a dataset may take on negative values.

It is important to note that, even when using a relative time scale, negative values only serve as labels for a position in time relative to a certain point in time and do not carry the same mathematical importance as negative numbers in other instances (Chatterjee, and Hadi, 2012). The regression model would continue to handle time as a continuous variable, with coefficients and predictions interpreted from the specified starting point.

Considering the following cases:

Case 1. When $t > \tau$ and $Ce^{\hat{\beta}t} = \hat{\alpha} + \hat{\beta}y$

The equation in system (11) is satisfied, which utilizes as follows

$$Ce^{\hat{\beta}t} = \hat{\alpha} + \hat{\beta}y$$

In this case, substitute the system 10.1 to y and simplify the right hand side

$$\hat{\alpha} + \hat{\beta} \left[y(\tau)e^{\hat{\beta}(t-\tau)} - \frac{\hat{\alpha}}{\hat{\beta}}(1 - e^{\hat{\beta}(t-\tau)}) \right]$$

By distributive property, we have

$$\begin{aligned} & [\hat{\alpha} + \hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)}] + \left[-(\hat{\alpha} + \hat{\beta})\frac{\hat{\alpha}}{\hat{\beta}}(1 - e^{\hat{\beta}(t-\tau)}) \right] \\ & [\hat{\alpha} + \hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)}] + \left\{ (-\hat{\alpha} - \hat{\beta}) \left[\frac{\hat{\alpha}}{\hat{\beta}}(1 - e^{\hat{\beta}(t-\tau)}) \right] \right\} \end{aligned}$$

Simplifying further

$$\begin{aligned} & \hat{\alpha} + \hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)} - \hat{\alpha} - \hat{\alpha}(1 - e^{\hat{\beta}(t-\tau)}) \\ & \hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)} - \hat{\alpha}(1 - e^{\hat{\beta}(t-\tau)}) \end{aligned}$$

Which gives the following

$$\begin{aligned} & \hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)} - \hat{\alpha}[1 - e^{\hat{\beta}(t-\tau)}] \\ & \hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)} = \hat{\alpha}[1 - e^{\hat{\beta}(t-\tau)}] \end{aligned}$$

Divide to both sides by $\hat{\beta}$

$$y(\tau)e^{\hat{\beta}(t-\tau)} = \frac{\hat{\alpha}}{\hat{\beta}}[1 - e^{\hat{\beta}(t-\tau)}]$$

Finally, obtaining the system (10.1)

$$y(\tau)e^{\hat{\beta}(t-\tau)} - \frac{\hat{\alpha}}{\hat{\beta}}[1 - e^{\hat{\beta}(t-\tau)}]$$

The first term $y(\tau)e^{\hat{\beta}(t-\tau)}$ is the transient or decaying part of the solution, while the second term $\frac{\hat{\alpha}}{\hat{\beta}}[1 - e^{\hat{\beta}(t-\tau)}]$ represents the steady state or equilibrium part of the solution. Comparing this with the left-hand side $Ce^{\hat{\beta}t}$, observed that the equation holds.

Case 2: When $t < \tau$ and $Ce^{\hat{\beta}t} = -(\hat{\alpha} + \hat{\beta}y)$

Utilizing the system

$$Ce^{\hat{\beta}t} = -(\hat{\alpha} + \hat{\beta}y)$$

Substitute the system (10.2) to y

$$\begin{aligned} & -(\hat{\alpha} + \hat{\beta}) \left[y(\tau)e^{\hat{\beta}(t-\tau)} + \frac{\hat{\alpha}}{\hat{\beta}}(e^{\hat{\beta}(t-\tau)} - 1) \right] \\ & (-\hat{\alpha} - \hat{\beta}) \left[y(\tau)e^{\hat{\beta}(t-\tau)} + \frac{\hat{\alpha}}{\hat{\beta}}(e^{\hat{\beta}(t-\tau)} - 1) \right] \end{aligned}$$

By distributive property,

$$[(-\hat{\alpha} - \hat{\beta})y(\tau)e^{\hat{\beta}(t-\tau)}] + \left[(-\hat{\alpha} - \hat{\beta}) \frac{\hat{\alpha}}{\hat{\beta}}(e^{\hat{\beta}(t-\tau)} - 1) \right]$$

Simplifying further

$$\begin{aligned} & -\hat{\alpha} - \hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)} + -(\hat{\alpha} - \hat{\alpha})[e^{\hat{\beta}(t-\tau)} - 1] \\ & -\hat{\alpha} - \hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)} - \hat{\alpha} + \hat{\alpha}[e^{\hat{\beta}(t-\tau)} - 1] \\ & -\hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)} - \hat{\alpha}[e^{\hat{\beta}(t-\tau)} - 1] \end{aligned}$$

Which yields the following

$$-\hat{\beta}y(\tau)e^{\hat{\beta}(t-\tau)} = \hat{\alpha}[e^{\hat{\beta}(t-\tau)} - 1]$$

Divide to both sides $-\hat{\beta}$

$$y(\tau)e^{\hat{\beta}(t-\tau)} = -\frac{\hat{\alpha}}{\hat{\beta}}[e^{\hat{\beta}(t-\tau)} - 1]$$

Finally, obtaining the system (10.2)

$$y(\tau)e^{\hat{\beta}(t-\tau)} + \frac{\hat{\alpha}}{\hat{\beta}}[e^{\hat{\beta}(t-\tau)} - 1]$$

Therefore, both cases are valid solutions for $\hat{y}(t)$ given the initial condition $y(\tau)$ and the parameters $\hat{\alpha}$ and $\hat{\beta}$ in the system (11), where $t > \tau$.

3.3 The Actual Parameters

To verify the theoretical result, the researcher shows the time series data plot of Unemployment Rate in Ilocos Region, Philippines quarterly from January 2018 to October 2023. This means that the quarterly rate of unemployment rate was plotted against time. The time series data plot shown in Figure 1 shows that between January 2018 and April 2019, the unemployment rate changed moderately. However, in April 2020, there was a considerable increase, reaching 22.3%, which was most likely due to the economic consequences of the COVID-19 pandemic, which resulted in big job losses and economic instability. According to Austria (2020), during the pandemic, the Department of Labor and Employment (DOLE), in collaboration with the Pangasinan Public Employment Services Office (PESO), recently held the first face-to-face (F2F) job fair, catering to around 300 Pangasinense job seekers. From July 2022 to October 2022, the unemployment rate dropped gradually showing a job market bounce back as economic activity restarted and economic stimulus strategies were implemented to help organizations and workers. This declining trend peaked in October 2022, when the unemployment rate reached its lowest point of 4%. Between January 2023 and October 2023, the unemployment rate remained stable, ranging between 4.4% and 4.7%. This stability reflects a time of job market equilibrium, in which the unemployment rate does not significantly increase or fall.

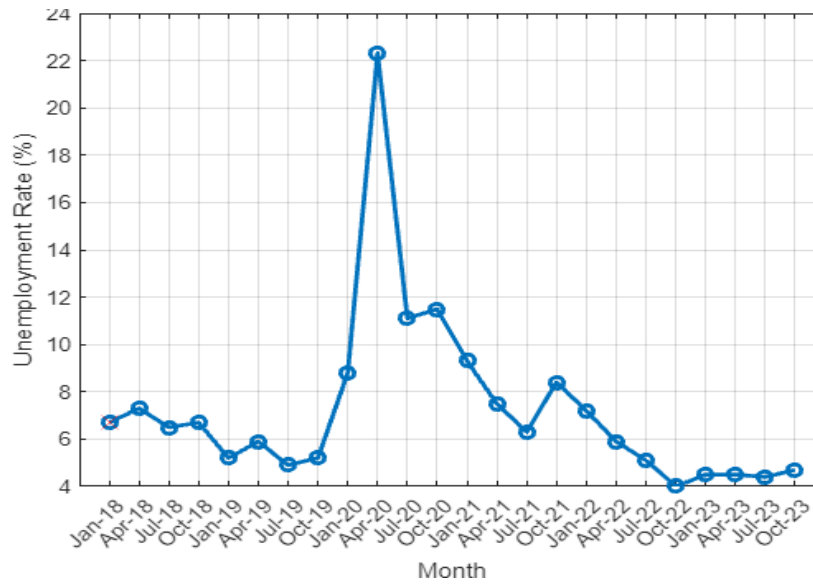


Figure 1. Time series data plot of unemployment rate in Ilocos Region

Table 1 presents the following values for the derivative of data points.

Table 1. Computations for derivative of data observations

Derivatives of Data Points			
y'_1	0.6	y'_{13}	-1.8
y'_2	-0.8	y'_{14}	-1.2
y'_3	0.2	y'_{15}	2.1
y'_4	-1.5	y'_{16}	-1.2
y'_5	0.7	y'_{17}	-1.3
y'_6	-1	y'_{18}	-0.8
y'_7	0.3	y'_{19}	-1.1
y'_8	3.6	y'_{20}	0.5
y'_9	13.5	y'_{21}	0
y'_{10}	-11.2	y'_{22}	-0.1
y'_{11}	0.4	y'_{23}	0.3
y'_{12}	-2.2	y'_{24}	-4.7

For computations in Table 1, the mean of the derivative of data observations $\bar{y}'$ is -0.2792 .

The y'_i is the rate of change of the data values with respect to time, calculated as the difference in consecutive data points divided by the corresponding difference in time intervals, y'_i quantifies how the values in the dependent are changing per unit time at the specific time point t_i (Kiusalaas, 2016), and used in the formulation of the loss function $J_{se}(\alpha, \beta)$ to measure the deviation between the estimated derivative and the actual, contributing to the overall process of estimating the parameters α and β that define the first-order linear ODE modeling the data.

Mean of the Observation Values Computation:

$$\bar{y} = \frac{173.9}{24} \approx 7.26$$

This means that $\bar{y} \approx 7.26$ reflects a large proportion of data values, serving as a reference point for interpreting the data's overall behavior. This intercept is the most important variable in the model since it represents the beginning point of the relationship between the variables (Mali, 2024). According to Mariotti (2023) this can help improve the model's performance and interpretability while also providing insights regarding its goodness of fit and the need for further development or adjustment. This implies that the intercept value can help determine the need for more model development to better fit the observed data and connection.

Mean of the Derivatives of Data Computation:

$$\bar{y}' = \frac{1}{n} \sum_{i=1}^n y'_i = \frac{-6.7}{24} \approx -0.2792.$$

This means that, on average, -0.2792 values in the dependent variable decrease in the data over time at a rate of around -0.2792 units per unit of time. The negative derivative is important for understanding the overall behavior of the system. According to Tedeshi (2023), this serves as crucial for understanding the underlying dynamics of the system being modeled and can help influence the selection of appropriate modeling techniques or assumptions. The data suggests the mean derivative value of -0.2792 serves as an initial point for this loss function, which is important to the overall process of estimating α and β , which represent the first-order linear ODE model of the data.

According to Solak (2022) observed function values and derivatives may have changing noise levels, which are compensated for by introducing diagonal contributions with adjusted hyper parameters. Inference and prediction are done as usual. This strategy was used to learn in dynamic systems.

Table 2 shows the minimization and computations for parameter $\hat{\beta}$, including the numerator and the denominator.

Table 2. Minimization and Computations for $\hat{\beta} = \frac{\sum_{i=0}^n (y_i)[(y'_i) - \bar{y}']}{\sum_{i=0}^n (y_i)(y_i - \bar{y})}$

Time (t)	Derivative of Data Observation	Numerator	Denominator
1	0.6	5.890439	-3.6570811
2	-0.8	-3.802059	0.3954191
3	0.2	3.114605	-4.8479145
4	-1.5	-8.179561	-3.6570811
5	0.7	5.091684	-10.6383316
6	-1	-4.252897	-7.9404147
7	0.3	2.837933	-11.4945817
8	3.6	20.171684	-10.6383316
9	13.5	121.256696	13.6766696
10	-11.2	-243.534509	335.7079241
11	0.4	7.538787	42.7812537
12	-2.2	-22.089545	48.9229205
13	-1.8	-14.143719	19.1037531
14	-1.2	-6.906225	1.9062525
15	2.1	14.988771	-5.9587479
16	-1.2	-7.734972	9.6950028
17	-1.3	-7.349976	-0.3299976
18	-0.8	-3.072897	-7.9404147
19	-1.1	-4.186233	-10.9437483
20	0.5	3.11668	-12.983332
21	0	1.256265	-12.3562485
22	-0.1	0.806265	-12.3562485
23	0.3	2.548348	-12.5216652
24	-4.7	-20.777901	-11.9654151
SUM	$\sum_{i=0}^n y'_i = -0.27917$	$\sum_{i=0}^n (y_i)[(y'_i) - \bar{y}'] = -157.412337$	$\sum_{i=0}^n (y_i)(y_i - \bar{y}) = 331.9596413$

$\hat{\beta}$ Computation:

$$\hat{\beta} = \frac{\sum_{i=0}^n (y_i)[(y'_i) - \bar{y}']}{\sum_{i=0}^n (y_i)(y_i - \bar{y})} = \frac{-157.412337}{331.9596413} \approx -0.474$$

This means that the value of $\hat{\beta} \approx -0.474$ represents the coefficient of the variable y in the first-order linear ordinary differential equation. A negative value for $\hat{\beta}$ indicates that the rate of change of y with respect to time is

inversely related to the current value of y (Boyce, 2024). In simpler terms, as y increases, the rate of change of y decreases, and vice versa.

$\hat{\alpha}$ Computation:

$$\hat{\alpha} = \bar{y}' - \hat{\beta}\bar{y} = -0.27917 - (-0.474)(7.245833) \approx 3.1568$$

This means that the value of $\hat{\alpha} \approx 3.1568$ corresponds to the constant term in the first-order linear ordinary differential equation. It indicates the starting point for the rate of change of y over time, regardless of y 's current value.

In the simulation result show that the general equation for the first-order linear ordinary differential equation model for the unemployment rate in Ilocos Region from January 2018 to October 2023 quarterly is

$$\frac{dy}{dt} = 3.1620 - 0.474y \quad (12)$$

and the specific solution of the first-order linear ODE is

$$\hat{y}(t) = y(\tau)e^{-0.474(t-\tau)} + \frac{3.1620}{0.474} [1 - e^{-0.474(t-\tau)}] \quad (13)$$

To verify the solution of the first-order linear ODE, the researcher proves the initial condition 6.7 at $\tau = 0$. The standard form of the system (4) for differential equations in a first-order linear ordinary differential equation

$$\frac{dy}{dt} + 0.496y = 3.1620$$

By using an integrating factor. The integrating factor is

$$e^{\int 0.474 dt} = e^{0.474t}$$

Multiply both sides of the differential equation by the integrating factor

$$e^{\int 0.474 dt} \frac{dy}{dt} + 0.474e^{\int 0.474 dt} y = 3.1620e^{0.474t}$$

Now, the left-hand side can be expressed as the derivative of the product of the integrating factor and the dependent variable y .

$$\frac{d}{dt} e^{\int 0.474 dt} y = 3.1620e^{0.474t}$$

Integrate both sides with respect to t

$$\int \frac{d}{dt} e^{\int 0.474 dt} y = \int 3.1620e^{0.474t} dt$$

By applying the formula $\int e^{at} = \frac{1}{a} e^{at} + C$, where $a = 0.474$. It follows that

$$e^{\int 0.474 dt} y = \frac{3.1620}{0.474} e^{0.474t} + C$$

Multiply to both sides of the equations $\frac{1}{e^{\int 0.474 dt}}$

$$y = \frac{1}{e^{0.474t}} (6.670970464e^{0.474t} + C)$$

Finally, obtaining

$$y(t) = 6.670970464 + Ce^{-0.474t}$$

Solving for the value of C when $t = 0$ yields to

$$y(0) = 6.670970464 + Ce^{-0.474(0)}$$

Substituting $y(0)$ to 6.7, gives

$$\begin{aligned} 6.7 &= 6.670970464 + Ce^{-0.474(0)} \\ 6.7 &= 6.670970464 + C \\ C &= 0.02902953586 \end{aligned}$$

Solving the initial condition $y(0)$, when $C = 0.02902953586$

$$\begin{aligned} y(0) &= 6.670970464 + 0.02902953586e^{-0.474(0)} \\ y(0) &= 6.670970464 + 0.02902953586 \\ y(0) &\approx 6.7 \end{aligned}$$

The theorem below presents the initial value, which gives us the solution for the first-order linear ordinary differential equation at the given time.

Theorem 4.

(Initial condition) Show that if the initial condition $y(0) = 6.7$, then $\tau = 0$ for $t > \tau$.

Proof:

Consider the solution of the first-order linear ODE in the system (13)

$$\hat{y}(t) = y(\tau)e^{-0.474(t-\tau)} + \frac{3.1620}{0.474} [1 - e^{-0.474(t-\tau)}]$$

Since $\hat{y}(0) = y(\tau) = 6.7$, this implies that

$$6.7 = 6.7e^{-0.474(t-\tau)} + 6.670970464[1 - e^{-0.474(t-\tau)}]$$

Substituting $e^{-0.474(t-\tau)}$ to x , leads to

$$6.7 = 6.7x + 6.670970464[1 - x]$$

Which can be simplified to

$$\begin{aligned} 6.7 &= 6.7x + 6.670970464 - 6.670970464x \\ 6.7 &= 6.670970464 - 0.02902953586x \\ 6.7 - 6.670970464 &= -0.02902953586x \\ -0.02902953586 &= -0.02902953586x \end{aligned}$$

Obtaining

$$x = 1$$

When $t = 0$, then having

$$e^{-0.474(0-\tau)} = 1$$

Applying the natural logarithmic to the both sides

$$\begin{aligned} -0.474(-\tau) &= \ln 1 \\ 0.474\tau &= \ln 1 \\ \tau &= 0 \end{aligned}$$

Hence, when $\tau = 0$, this relationship holds true as time passes with t moving forward the initial time point τ . To summarize the result when $t = 0$ for $t > \tau$, this means that, the situation where time starts from zero at some point beyond the initial observation time τ and observing the system at a moment in time strictly greater than τ .

The initial value for the solution of a first-order linear ODE if $t = 0$ and $\tau = 0$ is

$$y(0) = 6.7e^{-0.474(0-\tau)} + \frac{3.1620}{0.474} [1 - e^{-0.474(0-\tau)}] \quad (14)$$

Using the parameter values from the ODE solution, the researcher graphs the ODE system's direction fields using MatLab software. This means that the direction fields are required to explain the behavior of the system given such parameters.

Figure 2 shows the dynamics between the actual data and the solution for the first-order linear ordinary differential equation. Using MatLab, the researcher graphs both the direction field and the ODE solution, indicating the direction of all system solutions. Using the direction field, the researcher generated a solution curve with an initial condition of 6.7. As the initial point moves along a solution curve, observe how the relationship between x and y changes over time. Notice that the curve looks to be closed in the sense that the value of approximately 6.67089 remains stable as time passes. Using the ODE solution, the researcher graphs using the Gaussian process regression with adjusted prediction regression dynamics system's direction fields using MatLab software. This means that the direction fields optimize the regression parameters to achieve the best-fit behavior in the presence of fluctuation from the system's real data given such parameters.

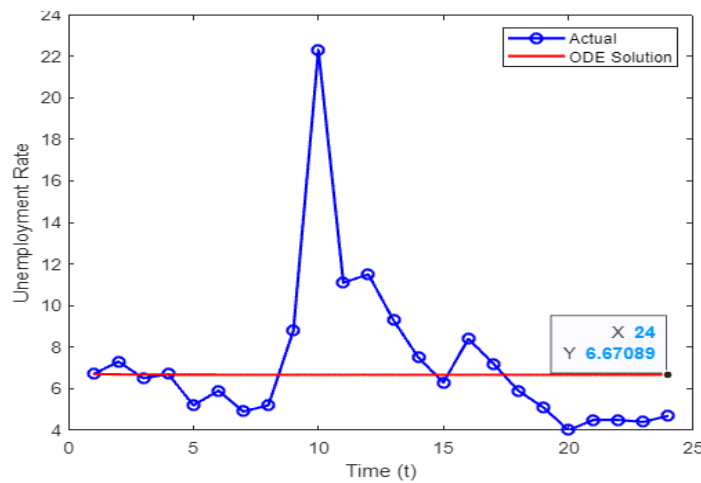


Figure 2. Actual data and solution for the first-order linear ODE of unemployment rate

Figure 3 reveals the behavior of the adjusted prediction with dynamic regression parameter direction fields. By adjusting the regression parameter based on dynamic changes in the unemployment rate, the adjusted values seek to increase model accuracy by considering changing dynamics over time. The GPR with adjusted prediction uses dynamic regression parameter dynamics to extract a more consistent trend from unemployment rate data, facilitating the detection of long-term trends while limiting the impact of short-term fluctuations.

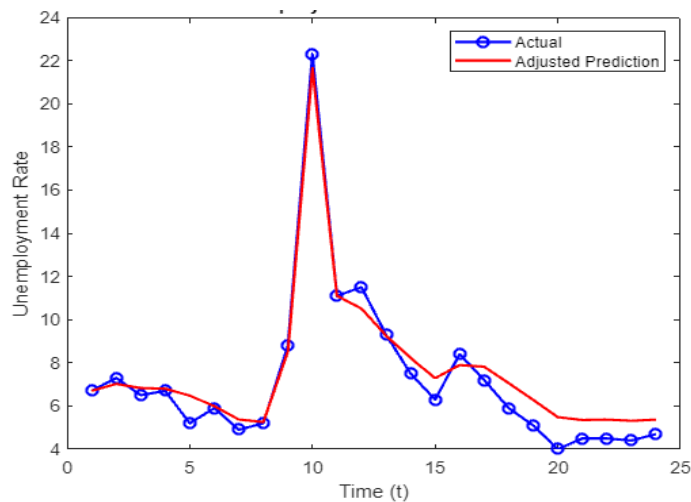


Figure 3. Actual data and adjusted prediction of unemployment rate

This means that the behavior of Gaussian Process Regression (GPR) with adjusted predictions can have a significant impact on unemployment rates in the Ilocos region. Policy effectiveness is how well the GPR model forecasts unemployment rates, which can help policymakers evaluate the efficacy of current regional unemployment-reduction measures. By comparing GPR forecasts to actual rates, policymakers can determine whether their policies are having the desired effect. The Early Warning System means that the GPR predictions can serve as an early warning system for potential increases in unemployment, allowing policymakers to proactively implement interventions or adjust policies to mitigate negative economic impacts. Investment planning means that businesses and investors can use GPR predictions to inform investment decisions in the Ilocos Region. Understanding future unemployment trends can help businesses anticipate labor market conditions and adjust their hiring and expansion plans accordingly. The social implications are that unemployment rates have social implications, including impacts on individual well-being, family stability, and community cohesion. Economic development means that the behavior of the GPR model can provide insights into the broader economic dynamics of the Ilocos Region. By understanding how unemployment rates fluctuate over time, policymakers can tailor economic development strategies to promote job creation and sustainable growth. Policy evaluation means that analyzing the relationship between GPR predictions and actual unemployment rates can facilitate the evaluation of past policy interventions. By assessing how well predictions align with outcomes, policymakers can identify areas where policies may need adjustment or improvement.

The economic relevance of the decrease in the unemployment rate over time suggests an improvement in the region's economic conditions. When unemployment decreases, it indicates that more people are finding jobs, leading to increased consumer spending, higher productivity, and overall economic growth. Gaussian Process Regression (GPR) with adjusted predictions is relevant in this context as it offers a statistical method to analyze and predict trends in data, such as the unemployment rate. By using historical data, GPR can model the relationship between time and the unemployment rate, enabling policymakers and economists to make more accurate forecasts and decisions. Policymakers can assess the effectiveness of their unemployment-reduction programs and actions by comparing GPR projections to actual unemployment rates. Furthermore, GPR can assist policymakers in identifying probable causes impacting unemployment patterns, allowing them to execute tailored measures to address specific labor market concerns. In conclusion, studying the unemployment rate in conjunction with GPR and adjusted forecasts provides useful insights into the region's economic trends. This enables more informed decision-making and policy formation to boost economic success and reduce unemployment.

The figures below represent the study's findings: a comparison of the ODE model and the modified adjusted prediction using dynamic regression.

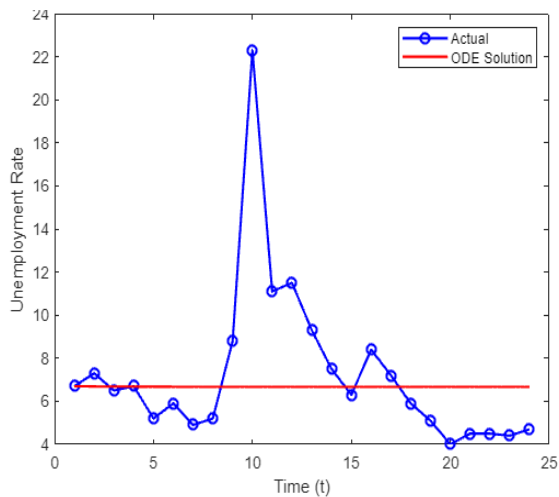


Figure 4a. Solution of the ODE

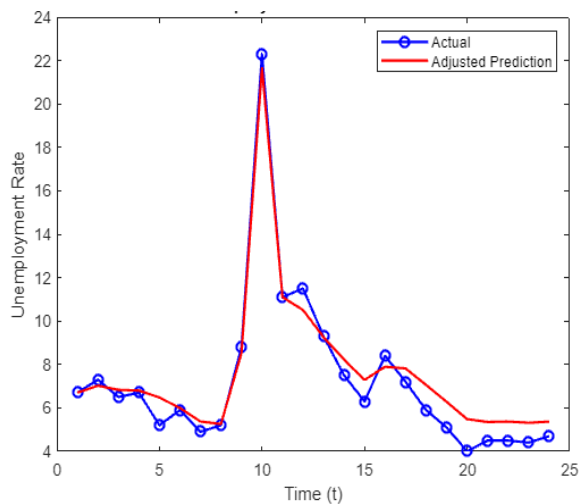


Figure 4b. Adjusted Prediction

Figure 4a shows the Ilocos Region's unemployment rate as well as the solution to the first-order linear ODE model. Prior to applying the modified regression ability, their values appeared to be different. The blue line represents the actual data curve, and the red line is the solution to the first-order linear ODE model, also known as the predicted curve. Figure 4b shows a satisfactory curve-fitting result, with the projected solution matching the actual data trend.

Computation for comparison of the root mean square error (RMSE) between the ODE solution and the adjusted prediction. The root mean squared error (RMSE) is a measure of the differences between values predicted by a model and the values actually observed.

RMSE ODE Solution Computation:

$$\text{RMSE} = \sqrt{\frac{339.8808617}{24}} \approx 3.7632$$

On average, the estimated values from the ODE solution are approximately 3.76 units off from the actual observed values. It means that the ODE solution makes an average error of approximately 3.76 units in predicting the observed values.

RMSE Adjusted Prediction Computation:

$$\text{RMSE} = \sqrt{\frac{13.37647664}{24}} \approx 0.74656$$

An RMSE of approximately 0.75 indicates that the modified predictions are greater than the actual observed values by only 0.75 units on average, indicating that the model fits the data accurately.

This means that the lower RMSE for the adjusted prediction model signifies improved prediction accuracy, better alignment with the observed data, and enhanced reliability for decision-making purposes. It indicates that the model, incorporating additional predictors, provides a more accurate representation of the relationship between variables and can make precise predictions.

4.0 Conclusion

The findings discussed in this study show that the researcher has made significant contributions to increasing our understanding of regression systems and their real-world applications. The comprehensive explanations for formulating regression equilibrium points and stability conditions will help readers better understand ordinary differential equations (ODEs) and regression systems. The researcher developed four theorems for analyzing parameter solutions such as $\hat{\beta}$ and $\hat{\alpha}$, as well as the regression model for actual data and initial conditions, to show the mathematical analysis's accuracy and completeness. Furthermore, using MATLAB simulations to check the model's stability and existence, as well as closely tracking actual data, demonstrate the suggested technique's real-world relevance and reliability. Furthermore, the availability of actual data demonstrates the relevance of mathematical modeling to real-world circumstances, indicating that the study has the potential to guide solutions to local challenges such as the Ilocos Region's unemployment rate.

The findings of this applied mathematics study have important implications for addressing economic challenges, particularly unemployment, in the Ilocos Region. The capacity to model and evaluate unemployment rate trends using regression systems and equilibrium instances can help policymakers and economists gain a better understanding of the underlying reasons for employment shifts, allowing them to design more targeted and successful interventions. The model's successful dynamic simulation, in which changed forecasts closely matched the actual data, implies that this modeling methodology can be used to anticipate and predict future unemployment patterns, allowing for the development of more informed economic plans and activities. Besides, identifying regression equilibrium points and conditions of stability can help determine the primary factors impacting the unemployment rate, allowing for more focused policy decisions and resource allocation to address the underlying reasons. Additionally, the interdisciplinary method, which combines mathematical

analysis, modeling tools, and real-world data, can be used to investigate and address a broader range of economic issues, including labor market dynamics, income distribution, and the impact of economic policies on various sectors. Overall, the findings show how rigorous mathematical modeling can provide useful insights and practical applications for bridging the gap between theoretical economic notions and evidence-based, successful economic policymaking.

The researcher's recommendations based on the conclusions obtained include several important aspects that focus on using the insights obtained from mathematical modeling and analysis of unemployment rate changes in the Ilocos Region to guide more effective authorities and economic actions. First, the researcher proposes that elected authorities and economists use their insight into the underlying reasons for employment trends to build specialized and focused plans to address the unemployment problem. Second, the successful simulation of the model, in which the updated forecasts nearly matched the actual data, indicates the modeling framework's ability to predict future unemployment trends, allowing for preventive economic planning and appropriate interventions. Third, choosing regression equilibrium points and stability needs can help determine the key variables impacting the unemployment rate, guide resource allocation, and guide the design of initiatives to address the core causes. Furthermore, the researcher suggests broadening the interdisciplinary approach by combining mathematical computations, modeling tools, and real-world data to investigate and address a broader range of economic issues other than unemployment, such as labor market dynamics, income distribution, and the impact of economic policies. Finally, the researcher underlines how mathematical modeling may bridge the gap between theoretical economic notions and actual, evidence-based policymaking, resulting in more informed and effective economic initiatives and activities.

5.0 Contributions of Authors

There is only one author for this research.

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7.0 Conflict of Interests

The author declares that they have no conflicts of interest.

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